

A Seminar on “Metric Space & Its Applications”

2017-18

By Sri Ajay Kumar Jena, Lecturer in Mathematics,
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ORGANISED BY DEPARTMENT OF MATHEMATICS, PATTAMUNDAI COLLEGE,
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REPORT

A seminar was organised by Department of Mathematics, Pattamundai College, Pattamundai on 28.10.2017 on the topic "**MATRIC SPACE & ITS APPLICATIONS**". Sri Ajay Kumar Jena , Lecturer in Mathematics, HNS College, Chandol who graced the seminar with his analytical thinking. We were able to get the beautiful glimpses of the students of our Department. Sri Arabinda Pandab, Head of the Department gave a key note address of the topic and welcomed the guests on the diace and the participants. The meeting was ended with a vote of thanks by Dr Nirmal Kumar Sahoo, another faculty member.

Metric Space & Its Applications

- Sri Ajay Kumar Jena, HNS College, Chandol

1. INTRODUCTION

Let X be an arbitrary set, which could consist of vectors in \mathbb{R}^n , functions, sequences, matrices, etc. We want to endow this set with a **metric**; i.e a way to measure distances between elements of X . A **distance** or **metric** is a function $d : X \times X \rightarrow \mathbb{R}$ such that if we take two elements $x, y \in X$ the number $d(x, y)$ gives us the distance between them. However, not just any function may be considered a metric: as we will see in the formal definition, a distance needs to satisfy certain properties.

Definition 1.1 (Metric Spaces). Given a set X and a function $d : X \times X \rightarrow \mathbb{R}$, we say that the pair $\mathcal{M} = (X, d)$ is a **metric space** if and only if $d(\cdot)$ satisfies the following properties:

- (1) (Non-negativeness) For all $x, y \in X$, $d(x, y) \geq 0$
- (2) (Identification) For all $x, y \in X$ we have that $d(x, y) = 0 \iff x = y$
- (3) (Symmetry) For all $x, y \in X$, $d(x, y) = d(y, x)$
- (4) (Triangular inequality) For all $x, y, z \in X$ we have that

$$(1.1) \quad d(x, z) \leq d(x, y) + d(y, z)$$

Property (1) just states that a distance is always a non-negative number. Property (2) tells us that the distance identifies points; i.e. if the distance between x and y is zero, it is because we are considering the same point. Property (3) states that a metric must measure distances symmetrically; i.e. it does not matter where we start measuring it. Finally, the triangular inequality is a generalization of the famous result that holds for the euclidean distance in the plane

2. EXAMPLES OF METRIC SPACES

2.1. Norms in vector spaces. Let $X = \mathbb{R}^n$. The typical distance used is the **euclidean distance**, defined as

$$d_2(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

For this metric, all properties except (1.1) are trivially shown. The triangular inequality is a well known result from linear algebra, known as the Cauchy-Schwartz inequality. However,

this is not the only distance we could define over \mathbb{R}^n . Consider the distance

$$d_1(x, y) \equiv \sum_{i=1}^n |x_i - y_i|$$

which is known as the “taxicab distance”. Another one we could use is the so called “max distance”, defined as

$$d_\infty(x, y) \equiv \max_{i \in [1:n]} |x_i - y_i|$$

where $[1:n] \equiv \{1, 2, \dots, n\}$. Notice that all this distances can be written as $d(x, y) = \mathbf{N}(x - y)$, for some function $\mathbf{N} : \mathbb{R}^n \rightarrow \mathbb{R}$. In general, one could define a lot of distances, based on different functions $N(\cdot)$ that one can come up with. In particular, sometimes we are interested in a subset of functions called **norms**. We will define them in a way that it applies to any **vector space** X (i.e. any space in which you can calculate $x + y$ and αx with $\alpha \in \mathbb{R}$ with the usual rules).

Definition 2.1 (Norms). Let X be a vector space (e.g. $X = \mathbb{R}^n$) and $\mathbf{N} : X \rightarrow \mathbb{R}$. We say $\mathbf{N}(\cdot)$ is a **norm** if the following 4 conditions hold:

- (i): $\mathbf{N}(x) \geq 0$ for all $x \in X$
- (ii): $\mathbf{N}(x) = 0 \iff x = 0$ for all $x \in X$
- (iii): $\mathbf{N}(\alpha x) = |\alpha| \mathbf{N}(x)$ for all $\alpha \in \mathbb{R}, x \in X$
- (iv): $\mathbf{N}(a + b) \leq \mathbf{N}(a) + \mathbf{N}(b)$ for all $a, b \in X$

Exercise 2.1. Show that the functions $\mathbf{N}_2(x) = \sqrt{x_i^2}$, $\mathbf{N}_1(x) = \sum_i |x_i|$ and $\mathbf{N}_\infty(x) = |x_i|$ are norms

Proposition 2.1. Let X be a vector space (e.g. $X = \mathbb{R}^n$) and define $d_{\mathbf{N}} : X \times X \rightarrow \mathbb{R}$ as

$$d_{\mathbf{N}}(x, y) \equiv \mathbf{N}(x - y)$$

Then, $\mathcal{M}_{\mathbf{N}} = (X, d_{\mathbf{N}})$ is a metric space

Proof. We need to prove each of the properties of a distance (from 1 to 4).

Non negativity (1) : Easy, since $\mathbf{N}(\cdot) \geq 0$ always

Identification (2) : Follows from

$$d_{\mathbf{N}}(x, y) = 0 \iff \mathbf{N}(x - y) = 0 \underbrace{\iff}_{(ii)} x - y = 0 \iff x = y$$

Symmetry (3) : Follows from

$$\begin{aligned} d_{\mathbf{N}}(x, y) &= \mathbf{N}(x - y) = \mathbf{N}((-1)(y - x)) \underbrace{=}_{(iii) \alpha=-1} \\ &= |-1| \mathbf{N}(y - x) = d_{\mathbf{N}}(y, x) \end{aligned}$$

Triangle inequality (4) : Take $x, y, z \in X$:

$$d_{\mathbf{N}}(x, z) = \mathbf{N}(x - y) = \mathbf{N}(x - z + z - y) \stackrel{\text{(iv)}}{\leq} \underbrace{\mathbf{N}(x - y)}_{=a} + \underbrace{\mathbf{N}(y - z)}_{=b} = d_{\mathbf{N}}(x, y) + d_{\mathbf{N}}(y, z)$$

□

2.2. Functional Spaces. Probably the most important new concept will be the space X that consists of functions instead of vectors. The most important one is the so-called “sup-norm metric” space: pick a norm $\mathbf{N} : \mathbb{R}^m \rightarrow \mathbb{R}$ and a set $A \subseteq \mathbb{R}^n$ and define

$$(2.1) \quad X = \{f : A \rightarrow \mathbb{R} \text{ such that } \exists K_f > 0 \text{ such that } \mathbf{N}[f(x)] \leq K_f \text{ for all } x \in A\}$$

$$(2.2) \quad d_{\infty}(f, g) \equiv \sup_{x \in A} \mathbf{N}(f(x) - g(x))$$

We define the metric space $\mathcal{M} \equiv \mathcal{B}(A, \mathbb{R}^m) = (X, d_{\infty})$ as the set of bounded functions from A to \mathbb{R}^m . Note that X is a vector space, defining the sum of functions as the point-wise sum; i.e.

$$(f + g)(x) = f(x) + g(x) \text{ for all } x \in A$$

since the sum of bounded functions is also bounded. Hence, if we show that the function

$$(2.3) \quad \mathbf{N}^*(f) \equiv \sup_{x \in A} |\mathbf{N}[f(x)]|$$

is itself a norm for X , we can apply Proposition 2.1 to show $\mathcal{B}(A, \mathbb{R}^m)$ is indeed a metric space. First, we need to show that $\mathbf{N}^*(f) < \infty$ for all $f \in X$ (i.e. it is a well defined object). This can be done since we are only taking functions that are bounded, and hence the supremum always exist (the so-called “Axiom of completeness” of the real numbers). Now, we show the properties of a norm.

(i) : $\mathbf{N}^*(f) \geq 0$ for all $f \in X$. This follows from \mathbf{N} being itself a norm

(ii) : $\mathbf{N}^*(f) = 0 \iff f(x) = 0$ for all $x \in A$.

$$\mathbf{N}^*(f) = 0 \iff \sup_{x \in A} \mathbf{N}[f(x)] = 0 \stackrel{\text{(a)}}{\iff} \underbrace{\mathbf{N}[f(x)] = 0}_{(a)} \text{ for all } x \in A \stackrel{\text{(b)}}{\iff} f(x) = 0 \text{ for all } x \in A$$

(a) follows from the fact that if $\mathbf{N}[f(\hat{x})] > 0$ for some $\hat{x} \in A$, then the sup would have to also be strictly positive. (b) follows from $\mathbf{N}(\cdot)$ being a norm (Property (ii))

(iii) : $\mathbf{N}^*(\alpha f) = |\alpha| \mathbf{N}^*(f)$. This comes from the homogeneity of the sup operator:

$$\mathbf{N}^*(\alpha f) = \sup_{x \in A} \mathbf{N}[\alpha f(x)] \stackrel{\text{(c)}}{=} \sup_{x \in A} |\alpha| \mathbf{N}[f(x)] = |\alpha| \sup_{x \in A} \mathbf{N}[f(x)] = |\alpha| \mathbf{N}^*(f)$$

where (c) follows from the fact that \mathbf{N} is a norm.

(iv) : *Triangular inequality.*

$$\begin{aligned} \mathbf{N}^*(f+g) &= \sup_{x \in A} \mathbf{N}[f(x) + g(x)] \leq \sup_{x \in A} \mathbf{N}[f(x)] + \mathbf{N}[g(x)] = \sup_{x, y \in A: x=y} \mathbf{N}[f(x)] + \mathbf{N}[g(y)] \leq \\ &\quad \sup_{x \in A} \mathbf{N}[f(x)] + \sup_{y \in A} \mathbf{N}[g(y)] = \mathbf{N}^*(f) + \mathbf{N}^*(g) \end{aligned}$$

and hence, $\mathcal{B}(A, \mathbb{R}^m)$ is indeed a metric space.

Exercise 2.2. Let $A \subseteq \mathbb{R}^n$. Show that the following are metric spaces: $\mathcal{L}_1(A, \mathbb{R}) = (X_1, d_1)$ where

$$\begin{aligned} X_1 &= \left\{ f : A \rightarrow \mathbb{R} \text{ such that } \int_{x \in A} |f(x)| dx < \infty \right\} \\ d_1(f, g) &\equiv \int_{x \in A} |f(x) - g(x)| dx \end{aligned}$$

and $\mathcal{L}_2(A, \mathbb{R}) = (X, d_2)$ where

$$X_1 = \left\{ f : A \rightarrow \mathbb{R} \text{ such that } \int_{x \in A} f^2(x) dx < \infty \right\}$$

$$d_2(f, g) \equiv \sqrt{\int_{x \in A} [f(x) - g(x)]^2 dx}$$

In general, show that given a norm $\mathbf{N} : \mathbb{R}^m \rightarrow \mathbb{R}$ the pair $\mathcal{L}_{\mathbf{N}}(A, \mathbb{R}) = (X_{\mathbf{N}}, d_{\mathbf{N}})$ defined as

$$\begin{aligned} X_{\mathbf{N}} &= \left\{ f : A \rightarrow \mathbb{R} \text{ such that } \int_{x \in A} \mathbf{N}[f(x)] dx < \infty \right\} \\ d_{\mathbf{N}}(f, g) &\equiv \int_{x \in A} \mathbf{N}[f(x) - g(x)] dx \\ \mathbf{N}^*(f) &= \left\{ \int_{x \in A} \mathbf{N}[f(x)]^\rho dx \right\}^{\frac{1}{\rho}} \end{aligned}$$

is a metric space as well, for any $\rho > 0$.

3. METRIC TOPOLOGY

In this section we will be studying the concept of “neighborhood” or closeness in generic metric spaces. This will be useful when generalizing concepts like “open sets”, “continuous functions”, “compact sets”, etc.

3.1. Open and closed balls.

Definition 3.1 (Open Ball). Given a metric space $\mathcal{M} = (X, d)$, $a \in X$ and $r > 0$ we define the open ball of center a and radius r as the set

$$B(a, r) = \{x \in X : d(a, x) < r\}$$



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Date.....28/10/17.....

To

Sri Ajay Kumar Jena
Lecturer in Mathematics,
HNS College, Chandol.

Sub: - An invitation as Resource Person in the Extramural Seminar in
Department of Mathematics.

Sir,

It is my pleasure to invite you as **Resource Person** in the Extramural Seminar on the topic "**Metric Space & Its Applications**" to be organized by Department of Mathematics, at 11.00 am on 28th October 2017 in our institution.

Your kind presence for this occasion is highly solicited.

Yours Faithfully,

Principal

Pattamundai College,
Pattamundai

Principal
Pattamundai College

An Extramural Seminar on "Metric Space & Its Applications"

Department of Mathematics

Pattamundai College, Pattamundai

Dt. 28.10.2017

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An Extramural Seminar on "Metric Space & Its Applications"

Department of Mathematics
Pattamundai College, Pattamundai

Dt. 28.10.2017

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